

q -FREQUENT HYPERCYCLICITY IN SPACES OF OPERATORS

MANJUL GUPTA AND ANEESH MUNDAYADAN

ABSTRACT. We provide conditions for a linear map of the form $C_{R,T}(S) = RST$ to be q -frequently hypercyclic on algebras of operators on separable Banach spaces. In particular, if R is a bounded operator satisfying the q -Frequent Hypercyclicity Criterion, then the map $C_R(S) = RSR^*$ is shown to be q -frequently hypercyclic on the space $\mathcal{K}(H)$ of all compact operators and the real topological vector space $\mathcal{S}(H)$ of all self-adjoint operators on a separable Hilbert space H . Further we provide a condition for $C_{R,T}$ to be q -frequently hypercyclic on the Schatten von Neumann classes $S_p(H)$. We also characterize frequent hypercyclicity of $C_{M_\varphi^*, M_\psi}$ on the trace-class of the Hardy space, where the symbol M_φ denotes the multiplication operator associated to φ .

1. INTRODUCTION

This paper is a continuation of our earlier work [14] on q -frequent hypercyclicity, which coincides with frequent hypercyclicity for $q = 1$. We study here this concept for linear maps defined on Banach algebras of operators on Banach and Hilbert spaces. Hypercyclicity in spaces of operators was initiated by K.C. Chan [7] and subsequently studied by J. Bonet, F. Martinez-Gimenez and A. Peris [3], K.C. Chan and R. Taylor [8], F. Martinez-Gimenez and A. Peris [17] and H. Petersson [19]. Indeed, left multiplication operators $\mathfrak{L}_R(S) = RS$ were considered in [3],[7],[8] and [17] and their general form $C_{R,T}(S) = RST$ was studied in [19]. A collective work in [3],[7] and [8] states that a bounded operator R on a separable Banach space X satisfies the Hypercyclicity Criterion if and only if the left multiplication operator \mathfrak{L}_R is hypercyclic on $\mathcal{L}(X)$ in the topology of pointwise convergence. This result holds for the topology of uniform convergence on compact subsets if X^* is separable and X has the approximation property, see [3]. In [19] H. Petersson proved that if T satisfies the Hypercyclicity Criterion in a separable Hilbert space, then \mathfrak{L}_T as well as the conjugate operator C_T is hypercyclic on the Schatten von Neumann classes $S_p(H)$, $1 \leq p < \infty$ and $\mathcal{K}(H)$.

In Section 3 we provide a sufficient criterion for $C_{R,T}$ to be q -frequently hypercyclic on the algebra of compact operators on Banach spaces and give applications to the unilateral and bilateral shift operators, and in Section 4 we continue the study in the space $S_p(H)$. Finally in Section 5, using an Eigenvalue Criterion, we characterize frequent hypercyclicity of certain maps of the form $C_{R,T}$ defined on spaces of operators on the classical Hardy space and ℓ^p .

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2. PRELIMINARIES

A continuous operator T on a topological vector space (TVS) X is said to be **hypercyclic** if the set $\{T^n x : n \geq 1\}$ is dense in X for some $x \in X$. For $q \in \mathbb{N}$ (*the set of natural numbers*), T is said to be **q -frequently hypercyclic** (see [14]) if there exists a vector $x \in X$ such that the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive q -lower density for each non-empty open set $U \subset X$, where the q -lower density of $A \subset \mathbb{N}$ is defined as

$$\underline{q\text{-dens}}(A) = \liminf_{N \rightarrow \infty} \frac{\text{card}\{n \in A : n \leq N^q\}}{N}.$$

For $q = 1$, the above notion is known as frequent hypercyclicity of an operator, studied in [1], [4] and [5]. If T is frequently hypercyclic, then it is q -frequently hypercyclic for all $q \in \mathbb{N}$, however, the converse is not true, cf. [14].

Let X and Y be separable Banach spaces. The space of all bounded (resp. of all compact) operators from X to Y is denoted by $\mathcal{L}(X, Y)$ (resp. $\mathcal{K}(X, Y)$). We shall use the symbols $\mathcal{L}(X)$ and $\mathcal{K}(X)$ for $\mathcal{L}(X, X)$ and $\mathcal{K}(X, X)$ respectively. The real subspace of $\mathcal{L}(H)$, of all self-adjoint operators on a separable infinite dimensional Hilbert space H is denoted by $\mathcal{S}(H)$ and is equipped with the topology of uniform convergence on compact subsets (COT). Also, for $p \in [1, \infty)$ the Schatten von Neumann class $S_p(H)$ is defined as the space of all operators $T \in \mathcal{L}(H)$ for which the approximation numbers $(a_n(T)) \in \ell^p$, where

$$a_n(T) = \inf\{\|T - F\| : \text{rank}(F) < n\}, n \geq 1,$$

See [10] and [11] for more details on the Schatten von Neumann classes.

For $R \in \mathcal{L}(X)$, the left and right multiplication operators are respectively defined as $\mathfrak{L}_R(S) = RS$ and $\mathfrak{R}_R(S) = SR$. Also, if R is a bounded operator on a Hilbert space, then the conjugate operator C_R is defined as $C_R(S) = RSR^*$.

Recall that a Banach space X is said to have the **approximation property (AP)** if the identity operator on X can be approximated by finite rank operators uniformly on compact subsets of X ; that is, for any $\epsilon > 0$ and $K \subset X$ compact, there exists an operator F of finite rank such that $\|F(x) - x\| < \epsilon$, $\forall x \in K$. If X has the AP, then finite rank operators are norm-dense in $\mathcal{K}(Y, X)$ for all Banach spaces Y , cf. [10] and [16].

A series $\sum_{n \geq 1} x_{n,j}$ in an F -space is said to be **unconditionally convergent uniformly** in $j \geq 0$ if for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\|\sum_{n \in F} x_{n,j}\| < \delta$ for all finite sets $F \subset [N, \infty)$ and j . We will make use of the following inequality in our subsequent work: let (λ_n) be a scalar sequence and $\sum_{n \geq 1} x_n$ a series in a Banach space. Then if $F \subset \mathbb{N}$ is finite, we have

$$(2.1) \quad \left\| \sum_{n \in F} \lambda_n x_n \right\| \leq 4 \sup_{n \in F} |\lambda_n| \sup_{G \subseteq F} \left\| \sum_{n \in G} x_n \right\|,$$

cf. [18] (See also [15]).

3. q -FREQUENT HYPERCYCLICITY IN $\mathcal{K}(X)$, $\mathcal{L}(X)$ AND $\mathcal{S}(H)$

In this section, we first obtain a sufficient criterion for the q -frequent hypercyclicity of $C_{R,T}$ on the Banach algebras $\mathcal{L}(X, Y)$ and $\mathcal{K}(X, Y)$, where $R \in \mathcal{L}(Y)$ and $T \in \mathcal{L}(X)$. The next result is already known for $q = 1$, cf. [13], Remark 9.10. However, following the proof of the frequent hypercyclicity criterion given in Theorem 6.18 of [2], we outline the proof for a given $q \in \mathbb{N}$.

Theorem 3.1. (*q -FHC Criterion*) *Let X be a separable F -space and D be a dense set in X . If for each $x \in D$, there exists a sequence $(x_n)_{n \geq 0}$ in X such that $x_0 = x$ and*

- (a) $\sum_{n=0}^r T^{r^q - (r-n)^q}(x)$ converges unconditionally, uniformly in $r \geq 0$,
- (b) $\sum_{n \geq 0} x_{(n+r)^q - r^q}$ converges unconditionally, uniformly in $r \geq 0$; and
- (c) $T^{n^q} x_{n^q} = x$, $T^{n^q} x_{m^q} = x_{m^q - n^q}$ for $m > n \geq 0$,

then T is q -frequently hypercyclic on X .

Proof. Without loss of generality, one may assume that D is countable. Write $D = \{x_k : k \in \mathbb{N}\}$ and fix (ϵ_k) such that $k\epsilon_k + \sum_{j \geq k+1} \epsilon_j \rightarrow 0$. For each x_k , there exists a sequence $(x_{n,k})_{n \geq 0}$ with the conditions in the hypotheses being satisfied. By (a) and (b), it is possible to find an increasing sequence (N_k) of natural numbers such that $\|\sum_{n \in F} T^{r^q - (r-n)^q}(x_i)\| < \epsilon_k$ for $F \subset [N_k, \infty) \cap \{1, \dots, r\}$, and $\|\sum_{n \in G} x_{(n+r)^q - r^q, i}\| < \epsilon_k$ for $G \subset [N_k, \infty)$, uniformly in $r \geq 0$, where $1 \leq i \leq k$. By Lemma 6.19 of [2], corresponding to (N_k) , we find a sequence (J_k) of subsets of \mathbb{N} such that $\underline{\text{dens}}(J_k) > 0$, $\min(J_k) \geq k$ and $|m - n| \geq N_k + N_j$ for all $m \in J_k$, $n \in J_j$ and $m \neq n$. With these properties, the vector $x = \sum_{\ell \geq 1} \sum_{n \in J_\ell} x_{n^q, \ell}$ is a frequently hypercyclic vector for the sequence (T^{n^q}) , and thus it is a q -frequently hypercyclic vector for T . \square

Let $y \otimes x^*$ be the one-rank operator $x \rightarrow x^*(x)y$, where $y \in Y$ and $x^* \in X^*$ and $(\mathcal{I}(X, Y), \|\cdot\|_{\mathcal{I}(X, Y)})$ be a Banach space of operators from X to Y such that the set of all finite-rank operators is $\|\cdot\|_{\mathcal{I}(X, Y)}$ -dense in $\mathcal{I}(X, Y)$ and $\|y \otimes x^*\|_{\mathcal{I}(X, Y)} = \|y\| \|x^*\|$ for all $y \in Y$ and $x^* \in X^*$. We have the following result concerning the separability of $\mathcal{L}(X, Y)$ with respect to the topologies SOT and COT, and of $\mathcal{K}(X, Y)$ in the operator norm topology.

Proposition 3.2. *Let X and Y be separable Banach spaces. Then the following are true.*

- (1) *If D is a countable dense subset of Y and Φ is a countable weak*-dense subset of X^* , then the set*

$$\mathcal{G}_{D, \Phi} = \left\{ \sum_{n=1}^N y_n \otimes x_n^* : y_n \in D, x_n^* \in \Phi, N \in \mathbb{N} \right\}$$

is a countable SOT-dense subset of $\mathcal{L}(X, Y)$.

- (2) *If X^* is separable and Φ is norm-dense, then the above set $\mathcal{G}_{D, \Phi}$ is $\|\cdot\|_{\mathcal{I}(X, Y)}$ -dense in $\mathcal{I}(X, Y)$.*
- (3) *Suppose that X^* is separable and Y has the AP. If Φ is norm-dense, then $\mathcal{G}_{D, \Phi}$ is norm-dense in $\mathcal{K}(X, Y)$ and COT-dense in $\mathcal{L}(X, Y)$.*

Proof. The proof of (1) is similar to the case of $X = Y$, proved in [13], p. 277. Further, by the properties of $\mathcal{I}(X, Y)$ mentioned above, part (2) follows since we can approximate every operator of

finite rank by elements of $\mathcal{G}_{D,\Phi}$ in the norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. To get part (3), let us assume that Y has the AP. Then the space $\mathcal{F}(X,Y)$ of all finite-rank operators is norm-dense in $\mathcal{K}(X,Y)$ for every Banach space X . Moreover, $\|v \otimes u^*\|_{op} = \|v\| \|u^*\|$ for all $v \in Y$ and $u^* \in X^*$. \square

Using the above proposition, we prove

Theorem 3.3. *Let $R \in \mathcal{L}(Y)$ and $T \in \mathcal{L}(X)$ for separable Banach spaces X and Y and $q \in \mathbb{N}$. Let \mathcal{D} be a norm-dense set in Y and Φ be a countable weak*-dense set in X^* . Suppose that for each $(y, x^*) \in \mathcal{D} \times \Phi$, there exist sequences $(y_n)_{n \geq 0}$ in Y and $(x_n^*)_{n \geq 0}$ in X^* such that*

- (a) *the series $\sum_{n=0}^r R^{r^q-(r-n)^q}(y) \otimes (T^*)^{r^q-(r-n)^q}(x^*)$ and $\sum_{n=1}^{\infty} y_{(n+r)^q-r^q} \otimes x_{(n+r)^q-r^q}^*$ are unconditionally convergent in $(\mathcal{L}(X,Y), \|\cdot\|_{op})$, uniformly in $r \geq 0$; and*
- (b) *$R^{n^q} y_{n^q} = y$, $(T^*)^{n^q} x_{n^q}^* = x^*$, $R^{n^q} y_{m^q} = y_{m^q-n^q}$, and $(T^*)^{n^q} x_{m^q}^* = x_{m^q-n^q}^*$ for all $m > n \geq 0$.*

Then the following assertions hold.

- (i) *If $T^*(\Phi) \subseteq \Phi$, $\{x_n^*\} \subseteq \Phi$, then $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{L}(X,Y), \text{SOT})$.*
- (ii) *If Y has the AP and the set Φ is norm-dense in X^* , then $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{K}(X,Y), \|\cdot\|_{op})$ and $(\mathcal{L}(X,Y), \text{COT})$.*

Proof. (i) As Y is a separable Banach space, we may assume that D is countable. Let

$$\mathcal{L}_{\Phi} = \overline{\text{span}}^{\|\cdot\|_{op}} \{y \otimes x^* : y \in Y, x^* \in \Phi\},$$

where the closure is taken in the operator norm $\|\cdot\|_{op}$. Then \mathcal{L}_{Φ} is a separable Banach space since the set

$$\mathcal{G}_{D,\Phi} = \left\{ \sum_{n=1}^N y_n \otimes x_n^* : y_n \in D, x_n^* \in \Phi, N \in \mathbb{N} \right\}$$

is countable and norm-dense in \mathcal{L}_{Φ} .

If $G = \sum_{j \leq N} y_j \otimes x_j^*$, then $C_{R,T}(G) = \sum_{j \leq N} R(y_j) \otimes T^*(x_j^*)$. As the map $C_{R,T}$ is continuous and $T^*(\Phi) \subseteq \Phi$, it follows that $C_{R,T}$ takes \mathcal{L}_{Φ} to itself.

To establish the q -frequent hypercyclicity of the operator $C_{R,T}$ in SOT, we first show that $C_{R,T}$ satisfies the conditions of Theorem 3.1 in the space \mathcal{L}_{Φ} . Let $F = \sum_{j=1}^k y_j \otimes x_j^* \in \mathcal{G}_{D,\Phi}$. For each (y_j, x_j^*) , let $(y_{j,n})$ and $(x_{j,n}^*)$ be some sequences, respectively in Y and Φ , as in the hypothesis. Then

$$F_n = \sum_{j \leq k} y_{j,n} \otimes x_{j,n}^* \in \mathcal{L}_{\Phi}, n \geq 0.$$

Therefore, by the assumption (a) of our theorem, both the series

$\sum_{n \leq r} (C_{R,T})^{r^q-(r-n)^q}(F) = \sum_{j \leq k} \sum_{n \leq r} R^{r^q-(r-n)^q}(y_j) \otimes (T^*)^{r^q-(r-n)^q}(x_j^*)$ and $\sum_{n \geq 0} F_{(n+r)^q-r^q} = \sum_{j \leq k} \sum_{n \geq 0} y_{j,(n+r)^q-r^q} \otimes x_{j,(n+r)^q-r^q}^*$ converge unconditionally in \mathcal{L}_{Φ} with respect to the operator norm, uniformly in $r \geq 0$.

Further, we have

$$\begin{aligned} (C_{R,T})^{n^q} F_{m^q} &= \sum_{j=1}^k R^{n^q} y_{j,m^q} \otimes (T^*)^{n^q} x_{j,m^q}^* \\ &= \begin{cases} \sum_{j=1}^k y_{j,m^q-n^q} \otimes x_{j,m^q-n^q}^*, & m > n \\ F, & m = n. \end{cases} \end{aligned}$$

by hypotheses. Thus $(C_{R,T})^{n^q} F_{m^q} = F_{m^q-n^q}$ if $m > n$ and $(C_{R,T})^{n^q} F_{n^q} = F, n \geq 0$. So $C_{R,T}$ is q -frequently hypercyclic on \mathcal{L}_Φ with respect to the operator norm topology. As $\mathcal{G}_{D,\Phi}$ is SOT-dense in $\mathcal{L}(X, Y)$ by Proposition 3.2, the operator $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{L}(X, Y), \text{SOT})$. This establishes part (i).

Let us now prove part (ii). If Y has the AP and the set Φ is norm-dense in X^* , then by Proposition 3.2(3), $\mathcal{G}_{D,\Phi}$ is dense in $\mathcal{K}(X, Y)$ with respect to the operator norm and so $\mathcal{L}_\Phi = \mathcal{K}(X, Y)$. Consequently, $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{K}(X, Y), \|\cdot\|_{op})$ and $(\mathcal{L}(X, Y), \text{COT})$. \square

For applications of Theorem 3.3, we require the following lemmas.

Lemma 3.4. *Let X and Y be Banach spaces. If $\sum_{n=1}^{\infty} u_{n,j}$ is unconditionally convergent in Y , uniformly in $j \geq 0$, and $\{u_{n,k}^*\} \subset X^*$ is such that $\{u_{n,k}^* : n \geq N_0, k \geq 1\}$ is norm-bounded for some $N_0 \in \mathbb{N}$, then $\sum_n u_{n,j} \otimes u_{n,k}^*$ is unconditionally convergent in $(\mathcal{L}(X, Y), \|\cdot\|_{op})$, uniformly in $j, k \in \mathbb{N}$.*

Proof. By definition, for a given $\epsilon > 0$, one can choose a natural number $N > N_0$ such that $\|\sum_{n \in F} u_{n,j}\| < \epsilon$ for every finite set $F \subset [N, \infty) \cap \mathbb{N}$ and all $j \geq 1$.

Let $M \in \mathbb{R}$ be such that $\|u_{n,k}^*\| \leq M \forall n \geq N_0, k \geq 1$. Using the inequality (2.1), it follows that

$$\left\| \sum_{n \in F} u_{n,k}^*(x) u_{n,j} \right\| \leq 4M \sup_{G \subseteq F} \left\| \sum_{n \in G} u_{n,j} \right\| < 4M\epsilon,$$

for $j, k \geq 1$ and $\|x\| \leq 1$. Thus

$$\left\| \sum_{n \in F} u_{n,j} \otimes u_{n,k}^* \right\|_{op} < 4M\epsilon.$$

\square

Lemma 3.5. *Let X and Y be Banach spaces. If $\sum_{n=1}^j u_{n,j}$ is unconditionally convergent in Y , uniformly in $j \in \mathbb{N}$, and $\{u_{n,k}^* : n \geq N_0, k \geq 1\}$ is norm-bounded in X^* for some $N_0 \in \mathbb{N}$, then the series $\sum_n^j u_{n,j} \otimes u_{n,k}^*$ is unconditionally convergent in $(\mathcal{L}(X, Y), \|\cdot\|_{op})$, uniformly in $j \in \mathbb{N}$.*

Recalling the q -FHC Criterion from Theorem 3.1, we prove the q -frequent hypercyclicity of the left multiplication operator $\mathfrak{L}_R(S) = RS$. This strengthens a result of A. Bonilla and K.-G. Grosse-Erdmann [5] about the SOT-frequent hypercyclicity of \mathfrak{L}_R .

Corollary 3.6. *Let X be a separable Banach space and $R \in \mathcal{L}(X)$ satisfy the q -FHC Criterion. Then the following hold.*

- (i) \mathfrak{L}_R is q -frequently hypercyclic on $(\mathcal{L}(X), SOT)$.
- (ii) If X^* is separable and X has the AP, then the \mathfrak{L}_R is q -frequently hypercyclic on $(\mathcal{K}(X), \|\cdot\|_{op})$ and $(\mathcal{L}(X), COT)$.

Proof. In Theorem 3.3, let $X = Y$ and T be the identity operator on X . Since X is separable, the dual X^* is weak*-separable. So, we can choose Φ to be any countable weak*-dense subset of X^* . Since R satisfies the q -FHC Criterion, we find a set D satisfying the conditions (a) and (b) of Theorem 3.1. That is, for $x \in D$, there exists (x_n) in X with $x_0 = x$ such that

$$\sum_{n \leq r} R^{r^q - (r-n)^q}(x) \text{ and } \sum_{n \geq 1} x_{(n+r)^q - r^q} \text{ are unconditionally convergent uniformly in } r \geq 0,$$

and

$$R^{n^q} x_{n^q} = x, \quad R^{n^q} x_{m^q} = x_{m^q - n^q}, \quad m > n, \quad n \geq 0.$$

Now we verify the condition (a) of Theorem 3.3. For each $x^* \in \Phi$, let $x_n^* = x$, $n \geq 0$. By Lemmas 3.4 and 3.5 we get (i).

To see (ii), take Φ as any norm-dense subset of X^* in Theorem 3.3 and apply Theorem 3.3(ii). \square

Similarly, one can prove the following results. We observe that if $\sum_n x_{n,j}$ is unconditionally convergent, uniformly in j , then there exists $N \in \mathbb{N}$ such that the set $\{x_{n,j} : n \geq N, j \geq 1\}$ is bounded.

Corollary 3.7. *Suppose X is a separable Banach space and $T \in \mathcal{L}(X)$. Then the following are true.*

- (1) Let Φ be a countable weak*-dense subset of X^* . Suppose that for each $x^* \in \Phi$, there exists (x_n^*) in Φ with properties that $x_0^* = x^*$, the series $\sum_{n \leq r} (T^*)^{r^q - (r-n)^q}(x^*)$ and $\sum_{n \geq 1} x_{(n+r)^q - r^q}^*$ are unconditionally convergent in $(X^*, \|\cdot\|)$, uniformly in r ; and $(T^*)^{n^q} x_{n^q}^* = x^*$, $(T^*)^{n^q} x_{m^q}^* = x_{m^q - n^q}^*$, $m > n$. If $T^*(\Phi) \subset \Phi$, then \mathfrak{R}_T is q -frequently hypercyclic on $(\mathcal{L}(X), SOT)$.
- (2) Assume that X^* is separable and X has the AP. If T^* satisfies the q -FHC Criterion, then \mathfrak{R}_T is q -frequently hypercyclic on $(\mathcal{K}(X), \|\cdot\|_{op})$ and $(\mathcal{L}(X), COT)$.

Proposition 3.8. *Let $R, T \in \mathcal{L}(X)$, where R satisfies the q -FHC Criterion, and let Φ be a countable weak*-dense set in X^* . If for each $f \in \Phi$, there exists a bounded (f_n) in X^* such that $f_0 = 0$ and the set $\{(T^*)^n(f) : n \geq 0\}$ is bounded; and $(T^*)^{n^q} f_{n^q} = f$, $(T^*)^{n^q} f_{m^q} = f_{m^q - n^q}$ for $m > n \geq 0$, then the following hold.*

- (1) If $T^*(\Phi) \subseteq \Phi$, then $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{L}(X), SOT)$.
- (2) If Φ is norm-dense, and X has the AP, then $C_{R,T}$ is q -frequently hypercyclic on $(\mathcal{K}(X), \|\cdot\|_{op})$ and $(\mathcal{L}(X), COT)$.

Let us now establish the q -frequent hypercyclicity of $C_{R,U}$ for a unitary U and of the conjugate operator $C_R(S) = RSR^*$.

Corollary 3.9. *Let R satisfy the q -FHC Criterion in a separable Hilbert space H . Then the operators C_R and $C_{R,U}$ are q -frequently hypercyclic on $(\mathcal{K}(H), \|\cdot\|_{op})$ and $(\mathcal{L}(H), COT)$, where $U \in \mathcal{L}(H)$ is a unitary operator.*

Proof. Since H is a separable Hilbert space, it has the AP. As in the proof of the above results, one can find a dense set D of H satisfying the conditions in Theorem 3.1. Now, in Proposition 3.8 take $T = R^*$ and $\Phi = D$. The result follows. \square

We now proceed to some concrete applications of Theorem 3.3. We provide sufficient conditions on the weights (w_n) and (μ_n) for the map C_{B_w, F_μ} to be q -frequently hypercyclic on different Banach algebras of operators on ℓ^p , $1 \leq p < \infty$, where the backward shift B_w and the forward shift F_μ are respectively given by $B_w(e_0) = 0$, $B_w(e_n) = w_n e_{n-1}$, $n \geq 1$ and $F_\mu(e_n) = \mu_{n+1} e_{n+1}$, $n \geq 0$. Here $\{e_n\}_{n \geq 0}$ is the standard basis in ℓ^p .

Proposition 3.10. *If $\lim_{n \rightarrow \infty} |w_1 w_2 \dots w_{(n+r)q-rq+i} \mu_1 \mu_2 \dots \mu_{(n+r)q-rq+j}| = \infty$, uniformly in $r \geq 0$, for all $i, j \in \mathbb{N}_0$, then C_{B_w, F_μ} is q -frequently hypercyclic on $(\mathcal{L}(\ell^1), SOT)$, $(\mathcal{K}(\ell^p), \|\cdot\|_{op})$ and $(\mathcal{L}(\ell^p), COT)$, where $1 < p < \infty$.*

Proof. In Theorem 3.3, let $X = Y = \ell^p$. To prove the result for $(\mathcal{L}(\ell^1), SOT)$, let us write Φ_0 for the linear span of $\{e_n^* : n \geq 0\}$ over rationals in ℓ^∞ and \mathcal{D} for $\text{span}\{e_n : n \geq 0\}$ in ℓ^1 . Consider the maps S_w and J_μ given by $S_w(e_n) = \frac{1}{w_{n+1}} e_{n+1}$ and $J_\mu(e_n^*) = \frac{1}{\mu_{n+1}} e_{n+1}^*$, $n \geq 0$. Note that $B_w S_w$ and $F_\mu^* J_\mu$ are identity operators, and

$$(3.1) \quad S_w^m(e_n) = \frac{1}{w_{n+1} w_{n+2} \dots w_{n+m}} e_{n+m} \text{ and } J_\mu^m(e_n^*) = \frac{1}{\mu_{n+1} \mu_{n+2} \dots \mu_{n+m}} e_{n+m}^*.$$

Put

$$\Phi = \bigcup_{j \geq 0} \Phi_j, \text{ where } \Phi_{j+1} = \bigcup_{n, k \geq 0} (F_\mu^*)^n J_\mu^k(\Phi_j), j \geq 0.$$

Then the set Φ is weak*-dense in ℓ^∞ as $\Phi_0 \subseteq \Phi$. Clearly Φ is countable. Further $F_\mu^*(\Phi) \subseteq \Phi$ and $J_\mu(\Phi) \subseteq \Phi$.

We only consider the series

$$\sum_{n=1}^r B_w^{r^q - (r-n)^q}(e_i) \otimes (F_\mu^*)^{r^q - (r-n)^q}(e_j^*) \text{ and } \sum_{n=1}^\infty S_w^{(n+r)^q - r^q}(e_i) \otimes J_\mu^{(n+r)^q - r^q}(e_j^*)$$

for $i, j \geq 0$. As $B_w^n(e_i) = 0$ for sufficiently large n and $r^q - (r-n)^q \geq n$, the first series converges unconditionally in the operator norm, uniformly in $r \geq 0$. For the latter series, it suffices to prove that $\sum_{n \geq 1} a_{n,r} e_{(n+r)q-rq+i} \otimes e_{(n+r)q-rq+j}^*$ converges unconditionally, uniformly in the operator norm if $\lim_{n \rightarrow \infty} |a_{n,r}| = 0$, uniformly in r . But this is immediate as, for $x = (x_n) \in \ell^p$, $1 \leq p < \infty$, we have

$$\begin{aligned} \left\| \sum_{n \in F} a_{n,r} e_{(n+r)q-rq+i} \otimes e_{(n+r)q-rq+j}^* \right\| &= \left\| \sum_{n \in F} a_{n,r} x_{(n+r)q-rq+j} e_{(n+r)q-rq+i} \right\| = \\ &= \left(\sum_{n \in F} |a_{n,r} x_{(n+r)q-rq+j}|^p \right)^{1/p} \leq \max_{n \in F} |a_{n,r}| \|x\|. \end{aligned}$$

Therefore C_{B_w, F_μ} is q -frequently hypercyclic on $(\mathcal{L}(\ell^1), SOT)$ by Theorem 3.3(i).

Since ℓ^p has the AP and the set Φ constructed above is norm-dense in $\ell^{p'}$, where $1 < p < \infty$, $1/p + 1/p' = 1$ and $\ell^{p'}$ is the dual of ℓ^p , the operator C_{B_w, F_μ} is q -frequently hypercyclic on $(\mathcal{K}(\ell^p), \|\cdot\|_{op})$ and $(\mathcal{L}(\ell^p), COT)$ by Theorem 3.3(ii). \square

Remark 1. It is evident from the proof of the above proposition that the result holds for any Banach sequence space E with the AP such that $\text{span}\{e_n : n \geq 0\}$, $\text{span}\{e_n^*\}$ over rationals are norm-dense in E , E^* respectively and $\sum_n (w_1 w_2 \dots w_{i+(n+r)q-rq} \mu_1 \mu_2 \dots \mu_{j+(n+r)q-rq})^{-1} e_{i+(n+r)q-rq} \otimes e_{j+(n+r)q-rq}^*$ converges unconditionally in the operator norm, uniformly in r for all $i, j \geq 0$.

Our next aim is to obtain the bilateral version of Proposition 3.10. For $a = (a_n)_{n \in \mathbb{Z}}$ of a bounded sequence of nonzero scalars, we define the bilateral backward shift T_a on the sequence space $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, as $T_a(e_n) = a_n e_{n-1}$ and the forward shift S_a as $S_a(e_n) = a_n e_{n+1}$, $n \in \mathbb{Z}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the standard basis in $\ell^p(\mathbb{Z})$. Then we have

Proposition 3.11. *Suppose that, for all $i, j \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} |a_1 a_2 \dots a_{(n+r)q-rq+i} b_1 b_2 \dots b_{(n+r)q-rq+j}| = \infty$ and $\lim_{n \rightarrow \infty} |a_i a_{i-1} \dots a_{i-rq+(r-n)q+1} b_j b_{j-1} \dots b_{j-rq+(r-n)q+1}| = 0$, uniformly in $r \in \mathbb{N}_0$. Then C_{T_a, S_b} is q -frequently hypercyclic on $(\mathcal{L}(\ell^1(\mathbb{Z})), SOT)$, $(\mathcal{K}(\ell^p(\mathbb{Z})), \|\cdot\|_{op})$ and $(\mathcal{L}(\ell^p(\mathbb{Z})), COT)$, $1 < p < \infty$.*

Proof. We apply Theorem 3.3. Choose $X = Y = \ell^p(\mathbb{Z})$, $\mathcal{D} = \text{span}\{e_n : n \in \mathbb{Z}\}$ and $\Phi_0 = \text{span}\{e_n^* : n \in \mathbb{Z}\}$ over rationals. Define the maps S and J as $S(e_n) = \frac{1}{a_{n+1}} e_{n+1}$ and $J(e_n^*) = \frac{1}{b_{n+1}} e_{n+1}^*$ for $n \in \mathbb{Z}$. Let

$$\Phi = \bigcup_{j \geq 0} \Phi_j, \text{ where } \Phi_{j+1} = \bigcup_{n, k \geq 0} S_b^{*n} J^k(\Phi_j), j \geq 0.$$

The set \mathcal{D} is norm-dense in $\ell^p(\mathbb{Z})$ for $p \in [1, \infty)$, Φ is weak*-dense in $\ell^\infty(\mathbb{Z})$ and norm-dense in $\ell^{p'}(\mathbb{Z})$, where $1/p + 1/p' = 1$ and $1 < p < \infty$. Moreover $T_a S$ is the identity operator on \mathcal{D} and $S_b^* J$ is the identity on Φ . As in Proposition 3.10, $J(\Phi) \subseteq \Phi$ and $S_b^*(\Phi) \subseteq \Phi$. Further for $n \geq 1$ and $i, j \in \mathbb{Z}$, we have

$$T_a^n(e_i) = a_i a_{i-1} \dots a_{i-n+1} e_{i-n}, (S_b^*)^n(e_j^*) = b_j b_{j-1} \dots b_{j-n+1} e_{j-n}^*,$$

$$S^n(e_i) = \frac{1}{a_{i+1} a_{i+2} \dots a_{i+n}} e_{i+n} \text{ and } J^n(e_j^*) = \frac{1}{b_{j+1} b_{j+2} \dots b_{j+n}} e_{j+n}^*.$$

Let $S_n = S^n$ and $J_n = J^n$. Proceeding as in the proof of Proposition 3.10, one can show that the series $\sum_{n \leq r} T_a^{r^q - (r-n)^q}(e_i) \otimes (S_b^*)^{r^q - (r-n)^q}(e_j^*)$ and $\sum_{n \geq 1} S_{(n+r)q-rq}(e_i) \otimes J_{(n+r)q-rq}(e_j^*)$ are unconditionally convergent in the operator norm, uniformly in $r \geq 0$, by the hypothesis. This proves part (1).

Also since $\ell^p(\mathbb{Z})$ has the AP, and the above set Φ is norm-dense in $\ell^{p'}(\mathbb{Z})$ ($1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$), Theorem 3.3(ii) yields (2). \square

For the next result, let \mathbb{C}^N be considered as a vector space over \mathbb{C} , and for $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$, let D_μ be the diagonal operator $D_\lambda(f_j) = \lambda_j f_j$ on \mathbb{C}^N with respect to the standard basis $\{f_1, f_2, \dots, f_N\}$ of \mathbb{C}^N , where $N \in \mathbb{N}$. Then we have

Proposition 3.12. *Suppose that $|\lambda_j| \geq 1$ for each $1 \leq j \leq N$.*

- (1) *If $\sum_{n \geq 1} \frac{1}{|\mu_1 \mu_2 \dots \mu_{(n+r)^q - r^q + i}|} < \infty$ uniformly in r , for each $i \geq 0$, then C_{D_λ, F_μ} is q -frequently hypercyclic on $(\mathcal{L}(\ell^1, \mathbb{C}^N), SOT)$.*
- (2) *If $1 < p < \infty$ and $\sum_{n \geq 1} \frac{1}{|\mu_1 \mu_2 \dots \mu_{(n+r)^q - r^q + i}|^p} < \infty$ uniformly in r , for all $i \geq 0$, then C_{D_λ, F_μ} is q -frequently hypercyclic on $(\mathcal{K}(\ell^p, \mathbb{C}^N), \|\cdot\|_{op})$ and $(\mathcal{L}(\ell^p, \mathbb{C}^N), COT)$.*

Proof. In Theorem 3.3, take $X = \ell^p$ and $Y = \mathbb{C}^N$. Consider the set Φ and the maps J_n as in the proof of Proposition 3.10. Choose $D = \mathbb{C}^N$ and $S_n = S_\lambda^n$, where $S_\lambda(f_j) = \frac{1}{\lambda_j} f_j$. For a fixed $i \geq 0$, the series $\sum_{n \leq r} D_\lambda^{r^q - (r-n)^q}(f_j) \otimes (F_\mu^*)^{r^q - (r-n)^q}(e_i)$ is clearly unconditionally convergent in the operator norm, uniformly in r . Since $S_\lambda^n(f_j) = \lambda_j^{-n} f_j$ and $|\lambda_j| \geq 1$, the set $\{S_n(f_j) : n = 0, 1, 2, \dots\}$ becomes bounded. Consequently, the series $\sum_{n \geq 1} S_{(n+r)^q - r^q}(f_j) \otimes J_{(n+r)^q - r^q}(e_i)$ converges unconditionally, uniformly in r by Lemma 3.4. \square

So far, we have considered applications of Theorem 3.3 to maps on Banach algebras of operators. Now we turn to the q -frequent hypercyclicity of the conjugate operator $C_R(S) = RSR^*$ defined on the real subspace $\mathcal{S}(H)$ of $\mathcal{L}(H)$, consisting of all self-adjoint operators on a separable Hilbert space H . H. Petersson [19] showed that if R satisfies the Hypercyclicity Criterion, then C_R is hypercyclic on the norm-closure of $\text{span}\{h \otimes h : h \in H\}$ over \mathbb{R} , and hence COT-hypercyclic on $\mathcal{S}(H)$. A standard application of the q -FHC criterion, using Lemmas 3.4 and 3.5 yield the following:

Proposition 3.13. *If $R \in \mathcal{L}(H)$ satisfies the q -FHC Criterion, then C_R is q -frequently hypercyclic on $(\mathcal{S}(H), \text{COT})$.*

4. q -FREQUENT HYPERCYCLICITY IN $S_p(H)$

In this section we provide a sufficient condition for $C_{R,T}$ to be q -frequently hypercyclic on $S_p(H)$ for a separable Hilbert space H . Let us begin with the q -frequent hypercyclicity of the left multiplication operator \mathfrak{L}_R on $S_p(H)$, $1 \leq p < \infty$, which is an easy application of the q -FHC Criterion.

Proposition 4.1. *Suppose that R is an operator satisfying the q -Frequent Hypercyclicity Criterion in a Banach space X with separable dual.*

- (1) *If $(\mathcal{I}(X), \|\cdot\|_{\mathcal{I}})$ is a separable Banach ideal in $\mathcal{L}(X)$ such that the finite rank operators on X are $\|\cdot\|_{\mathcal{I}}$ -dense in $\mathcal{I}(X)$, and $\|x \otimes x^*\|_{\mathcal{I}} = \|x\| \|x^*\|$ for all $x \in X$ and $x^* \in X^*$, then the left multiplication operator \mathfrak{L}_R is q -frequently hypercyclic on $\mathcal{I}(X)$.*
- (2) *If X is a separable Hilbert space, then \mathfrak{L}_R is frequently hypercyclic on $S_p(X)$, $1 \leq p < \infty$.*

Proof. The proof of (1) is omitted. To see that (2) is true, recall that $\|u \otimes v\|_p = \|u\| \|v\|$ for all $u, v \in X$ and the finite rank operators on X form a dense subspace of $S_p(X)$. \square

For the main theorem of this section, we state the following result on summability of a series in $S_p(H)$, cf. [6], p. 152.

Lemma 4.2. Let $\{T_n\}_{n=1}^\infty \subset \mathcal{L}(H)$ be such that $T_n^* T_m = T_n T_m^* = 0$ whenever $m \neq n$. Then for $1 \leq p < \infty$

$$\|\sum_n T_n\|_p^p = \sum_n \|T_n\|_p^p.$$

We prove:

Theorem 4.3. Let $1 \leq p < \infty$, $R, T \in \mathcal{L}(H)$ and $D_1, D_2 \subset H$. Let D_1 and D_2 both span dense subspaces of H . If for each $(x, y) \in D_1 \times D_2$, there exist sequences $(x_n, y_n) \in H \times H$ with $(x_0, y_0) = (x, y)$ and

- (a) $\sum_{n=1}^r \|R^{r^q-(r-n)^q}(x)\|^p \|(T^*)^{r^q-(r-n)^q}(y)\|^p < \infty$ and $\sum_{n=1}^\infty \|x_{(n+r)^q-r^q}\|^p \|y_{(n+r)^q-r^q}\|^p < \infty$ uniformly in $r \geq 0$,
- (b) $\langle R^n(x), R^m(x) \rangle = \langle S_n(x), S_m(x) \rangle = \langle (T^*)^n(y), (T^*)^m(y) \rangle = \langle J_n(y), J_m(y) \rangle = 0$, for $m \neq n$; and
- (c) $R^{n^q} x_{n^q} = x$, $(T^*)^{n^q} y_{n^q} = y$, $R^{n^q} x_{m^q} = x_{m^q-n^q}$, $(T^*)^{n^q} y_{m^q} = y_{m^q-n^q}$, $\forall m > n \geq 0$,

then $C_{R,T}$ is q -frequently hypercyclic on $(S_p(H), \|\cdot\|_p)$.

Proof. Let $\Delta = \text{span}\{x \otimes y : x \in D_1, y \in D_2\}$. Note that Δ can also be written as the span of the set $\{x \otimes y : x \in \text{span} D_1, y \in \text{span} D_2\}$. Since $\text{span} D_1$ and $\text{span} D_2$ are dense in H , it can be proved that Δ is dense in $S_p(H)$, $1 \leq p < \infty$.

Let $F = \sum_{k=1}^N a_k x_k \otimes y_k \in \Delta$. Corresponding to x_k and y_k , we obtain sequences $(x_{k,n})$ and $(y_{k,n})$ as in the hypothesis. Set $F_n = \sum_{k=1}^N a_k x_{k,n} \otimes y_{k,n}$. Consider the series $\sum_{n \leq r} C_{R,T}^{r^q-(r-n)^q}(F) = \sum_{k=1}^N a_k \sum_{n=1}^\infty R^{r^q-(r-n)^q}(x_k) \otimes (T^*)^{r^q-(r-n)^q}(y_k)$ and $\sum_n F_{(n+r)^q-r^q} = \sum_{k=1}^N a_k \sum_{n=1}^\infty x_{k,(n+r)^q-r^q} \otimes y_{k,(n+r)^q-r^q}$. It suffices to prove that $\sum_{n \leq r} R^{r^q-(r-n)^q}(x) \otimes (T^*)^{r^q-(r-n)^q}(y)$ and $\sum_n x_{k,(n+r)^q-r^q} \otimes y_{k,(n+r)^q-r^q}$ are unconditionally convergent in $S_p(H)$, uniformly in r , for all $x \in D_1$ and $y \in D_2$.

Write $T_{n,r} = R^{r^q-(r-n)^q}(x) \otimes (T^*)^{r^q-(r-n)^q}(y)$, $n \geq 1$. Then $T_{n,r}^* = (T^*)^{r^q-(r-n)^q}(y) \otimes R^{r^q-(r-n)^q}(x)$. If $\langle \cdot, \cdot \rangle$ is the inner product in H , then $T_{n,r}^*(z) = \langle z, R^{r^q-(r-n)^q}(x) \rangle (T^*)^{r^q-(r-n)^q}(y)$ and $T_{m,r}(z) = \langle z, (T^*)^{r^q-(r-m)^q}(y) \rangle R^{r^q-(r-m)^q}(x)$ and so

$$T_{n,r}^* T_{m,r}(z) = \langle z, (T^*)^{r^q-(r-m)^q}(y) \rangle \langle R^{r^q-(r-m)^q} x, R^{r^q-(r-n)^q} x \rangle (T^*)^{r^q-(r-n)^q}(y).$$

Similarly

$$T_{n,r} T_{m,r}^*(z) = \langle z, R^{r^q-(r-m)^q} x \rangle \langle (T^*)^{r^q-(r-m)^q} y, (T^*)^{r^q-(r-n)^q} y \rangle R^{r^q-(r-n)^q} x.$$

From part (b) in the hypotheses, we get $T_{n,r}^* T_{m,r} = T_{n,r} T_{m,r}^* = 0$, $m \neq n$. Since $\|u \otimes v\|_p = \|u\| \|v\|$ for all $u, v \in H$, Lemma 4.2 and the hypothesis (a) yield that $\sum_{n \leq r} R^{r^q-(r-n)^q}(x) \otimes (T^*)^{r^q-(r-n)^q}(y)$ is unconditionally convergent in $S_p(H)$, uniformly in r . Similarly, one can obtain that $\sum_n x_{k,(n+r)^q-r^q} \otimes y_{k,(n+r)^q-r^q}$ is unconditionally convergent in $S_p(H)$, uniformly in r . Thus the condition (a) of Theorem 3.3 is satisfied by $C_{R,T}$ in $S_p(H)$. \square

Next, from Theorem 4.3, we obtain conditions on the weight sequences (w_n) and (μ_n) that are sufficient for the C_{B_w, F_μ} on $S_p(\ell^2)$ and C_{T_a, S_b} on $S_p(\ell^2(\mathbb{Z}))$ to be q -frequently hypercyclic, $1 \leq p < \infty$.

Proposition 4.4. (1) If $\sum_{n=1}^{\infty} |w_1 w_2 \dots w_{(n+r)q-rq+i} \mu_1 \mu_2 \dots \mu_{(n+r)q-rq+j}|^{-p} < \infty$ uniformly in $r \geq 0$ for all $i, j \geq 0$, then C_{B_w, F_μ} is q -frequently hypercyclic on $S_p(\ell^2)$.

(2) If $\sum_{n=1}^{\infty} |a_1 a_2 \dots a_{(n+r)q-rq+i} b_1 b_2 \dots b_{(n+r)q-rq+j}|^{-p} < \infty$ and $\sum_{n=0}^r |a_i a_{i-1} \dots a_{i-rq+(r-n)q+1} b_j b_{j-1} \dots b_{j-rq+(r-n)q+1}|^p < \infty$ uniformly in $r \geq 0$ for all $i, j \in \mathbb{Z}$, then C_{T_a, S_b} is q -frequently hypercyclic on $S_p(\ell^2(\mathbb{Z}))$.

Proof. To prove part (1), choose $D_1 = D_2 = \{e_n : n \geq 0\}$, the standard orthonormal basis in ℓ^2 . Let S_n and J_n be the maps as considered in the proof of Proposition 3.10, i.e., $S_n(e_i) = S_w^n(e_i) = \frac{1}{w_{i+1} \dots w_{i+n}} e_{i+n}$ and $J_n(e_j) = J_\mu^n(e_j) = \frac{1}{\mu_{j+1} \dots \mu_{j+n}} e_{j+n}$. Note that $\langle S_n(e_j), S_m(e_j) \rangle = \langle J_n(e_j), J_m(e_j) \rangle = 0$ for $n \neq m$. As $B_w^n(e_i) = 0$ for sufficiently large n , we have that $\sum_{n \leq r} \|B_w^{r^q-(r-n)^q}(e_i)\|^p \|(F_\mu^*)^{r^q-(r-n)^q}(e_j)\|^p < \infty$ uniformly in r . Moreover, from the hypothesis, we have $\sum_n \|S_{(n+r)q-rq}(e_i)\|^p \|J_{(n+r)q-rq}(e_j)\|^p < \infty$ uniformly in r . Now Theorem 4.3 yields the result.

To prove part (2), consider the maps S_n and J_n in the proof of Proposition 3.11 and choose $D_1 = D_2 = \{e_n : n \in \mathbb{Z}\}$, the standard orthonormal basis in $\ell^2(\mathbb{Z})$. Then, for $i, j \in \mathbb{Z}$, we have

$$S_n(e_i) = S^n(e_i) = \frac{1}{a_{i+1} a_{i+2} \dots a_{i+n}} e_{i+n} \text{ and } J_n(e_j) = J^n(e_j) = \frac{1}{b_{j+1} b_{j+2} \dots b_{j+n}} e_{j+n},$$

and

$$T_a^n(e_i) = a_i a_{i-1} \dots a_{i-n} e_{i-n-1} \text{ and } (S_b^*)^n(e_j) = b_j b_{j-1} \dots b_{j-n} e_{j-n-1}.$$

It follows by the hypotheses that the series $\sum_{n \geq 1} \|S_{(n+r)q-rq}(e_i)\|^p \|J_{(n+r)q-rq}(e_j)\|^p$ as well as

$$\sum_{n \leq r} \|T_a^{r^q-(r-n)^q}(e_i)\|^p \|(S_b^*)^{r^q-(r-n)^q}(e_j)\|^p \text{ converges uniformly } r \geq 0. \quad \square$$

5. FREQUENT HYPERCYCLICITY IN $S_p(H^2(\mathbb{D}))$ AND $\mathcal{N}(\ell^p)$

This section includes results on frequent hypercyclicity of specific operators of the form $C_{R,T}$ defined on the p th Schatten von-Neumann class of operators on the Hardy space $H^2(\mathbb{D})$, $1 \leq p < \infty$, as well as on the space $\mathcal{N}(\ell^p)$ of all nuclear operators on ℓ^p , $1 < p < \infty$. Let us recall the **Eigenvalue Criterion**, due to S. Grivaux.

Proposition 5.1. [12] *Let X be a separable, complex Banach space and $T \in \mathcal{L}(X)$. If for every countable subset D of the unit circle S^1 , the set $\bigcup_{\alpha \in S^1 \setminus D} \text{Ker}(T - \alpha I)$ spans a dense subspace of X , then T is frequently hypercyclic.*

Corresponding to a sequence $\beta = (\beta_n)$, $\beta_n > 0$, $n \geq 0$, let $(H^\beta(\mathbb{D}), \langle \cdot, \cdot \rangle)$ be a Hilbert space of complex functions, analytic on the open unit disc \mathbb{D} such that the evaluation mappings $f \rightarrow f(z)$ are continuous at each $z \in \mathbb{D}$, i.e. there exists $k_z \in H^\beta(\mathbb{D})$ such that $f(z) = \langle f, k_z \rangle$ for each $f \in H^\beta(\mathbb{D})$. Such a function k_z is called a **reproducing kernel** at $z \in \mathbb{D}$. Also, assume that $\{e_n\}_{n=0}^\infty$ forms an orthonormal basis for $H^\beta(\mathbb{D})$, where $e_n(z) = \beta_n z^n$. Note that when $\beta_n = 1$ for all $n \geq 0$, we have the Hardy space $H^2(\mathbb{D})$.

Let M_φ be the multiplication operator $f(z) \rightarrow \varphi(z)f(z)$ on $H^\beta(\mathbb{D})$, corresponding to an analytic function φ on \mathbb{D} and M_φ^* be the Hilbert space adjoint of M_φ . Our aim is to establish the frequent hypercyclicity of $C_{M_\varphi^*, M_\psi}$ on $S_p(H^\beta(\mathbb{D}))$, where M_φ and M_ψ are bounded multiplication operators on $H^\beta(\mathbb{D})$ corresponding to the analytic functions φ and ψ on \mathbb{D} . Let us first prove

Lemma 5.2. *Let φ and ψ be non-zero analytic functions on \mathbb{D} such that at least one of them is non-constant and $|\varphi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$. Then*

$$\text{span } \{k_z \otimes k_w : \overline{\varphi(z)}\psi(w) \in S^1 \setminus D\}$$

is dense in the space $S_1(H^\beta(\mathbb{D}))$ for every countable set $D \subset S^1$.

Proof. We write $\mathcal{H} = H^\beta(\mathbb{D})$ and $\overline{\varphi(\mathbb{D})} = \{\overline{\varphi(z)} : z \in \mathbb{D}\}$. By the open mapping theorem for analytic functions, the set $\overline{\varphi(\mathbb{D})}\psi(\mathbb{D}) = \{\overline{\varphi(z)}\psi(w) : z, w \in \mathbb{D}\} = \bigcup_{z \in \mathbb{D}} (\overline{\varphi(z)}\psi(\mathbb{D}))$ is non-empty and open. Hence there exists an open arc Γ in S^1 such that $\Gamma \subset \overline{\varphi(\mathbb{D})}\psi(\mathbb{D})$; let us assume that this arc Γ is the maximal one.

Consider the set

$$U \times V = \{(z, w) \in \mathbb{D} \times \mathbb{D} : \overline{\varphi(z)}\psi(w) \in \Gamma \setminus D\}.$$

We claim that U is uncountable, and for each $z \in U$, there exists an uncountable set $V_1 \subseteq V$ such that

$$(5.1) \quad \overline{\varphi(z)}\psi(w) \in \Gamma \setminus D, \text{ for all } w \in V_1.$$

To prove this, assume that both φ and ψ are non-constant. In this case, $\overline{\varphi(\mathbb{D})}$ is a non-empty open set. If $|\overline{\varphi(z)}\psi(w)| = 1$ for some $(z, w) \in \mathbb{D} \times \mathbb{D}$, then $\psi(w)\overline{\varphi(\mathbb{D})}$ is non-empty and open and so, we can find an arc $\Gamma_1 \subset \Gamma$ such that $\Gamma_1 \subset \psi(w)\overline{\varphi(\mathbb{D})}$ as above. Since D is countable and $\Gamma_1 \setminus D \subset \psi(w)\overline{\varphi(\mathbb{D})}$, the set U has to be uncountable. Now fix $z \in U$. Then the set $\overline{\varphi(z)}\psi(\mathbb{D})$, being non-empty and open, contains $\Gamma_2 \setminus D$ for some sub-arc Γ_2 of Γ . This proves the claim when φ and ψ are non-constant. Now assume that ψ is constant, say $\psi = c$ and φ is non-constant. We can proceed as above to prove that the set U is uncountable since $\overline{c\varphi(\mathbb{D})}$ is a non-empty open set containing $\Gamma \setminus D$. Also, since ψ is constant, we can take $V_1 = V = \mathbb{D}$. Finally when φ is constant and ψ is non-constant, we proceed similarly to get the result. Hence our claim is established.

We now show that $\Lambda = \text{span}\{k_z \otimes k_w : z \in U, w \in V\}$ is dense in the space $S_1(H)$. Recall that the trace of $A \in S_1(\mathcal{H})$ is given by $\text{tr}(A) = \sum_{n \geq 0} \langle Ae_n, e_n \rangle$, where $e_n(z) = \beta_n z^n$, $n \geq 0$. Also we have $S_1(H)^* = \mathcal{L}(H)$ with respect to the duality-pairing $(A, T) = \text{tr}(AT)$, $T \in S_1(H)$ and $A \in \mathcal{L}(H)$.

Let $A \in \mathcal{L}(H)$ be such that $\text{tr}(AT) = 0$ for all $T \in \Lambda$. For $T = k_z \otimes k_w$, we have $Te_n = \langle e_n, k_w \rangle k_z =$

$e_n(w)k_z = \beta_n w^n k_z$ and

$$\begin{aligned} \text{tr}(AT) &= \sum_{n \geq 0} \langle AT e_n, e_n \rangle \\ &= \sum_{n \geq 0} \beta_n w^n \langle k_z, A^* e_n \rangle \\ &= \sum_{n \geq 0} \beta_n \overline{(A^* e_n)(z)} w^n. \end{aligned}$$

Since $AT \in S_1(H)$, the above power series is well-defined for all $z, w \in \mathbb{D}$. Hence it is an analytic function in the variable w for a fixed $z \in \mathbb{D}$. For $z \in U$, there exists an uncountable set V_1 such that (5.1) holds. Since V_1 is uncountable, it has a limit point in \mathbb{D} . As $\beta_n > 0$ for all $n \geq 0$ and $\text{tr}(AT) = 0$, it follows that the coefficients of the above power series are all zero, i.e., $A^*(e_n)(z) = 0$ for all $n \geq 0$. Similarly, since $z \in U$ is arbitrary and U is uncountable, we have $A^*(e_n) = 0$, $\forall n \geq 0$. As $\{e_n : n \geq 0\}$ spans a dense subspace of \mathcal{H} , we conclude that $A = 0$. Therefore the set Λ is dense in $S_1(\mathcal{H})$. The proof is now complete. \square

The above lemma yields

Theorem 5.3. *Let φ and ψ be non-zero analytic functions on \mathbb{D} such that the corresponding multiplication operators are bounded on $H^\beta(\mathbb{D})$. If one of the maps φ and ψ is non-constant and $|\varphi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$, then $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic on $(S_p(H^\beta(\mathbb{D})), \|\cdot\|_p)$, $(\mathcal{K}(H^\beta(\mathbb{D})), \|\cdot\|_{op})$, and $((\mathcal{L}(H^\beta(\mathbb{D})), COT))$.*

Proof. For $z, w \in \mathbb{D}$, consider $k_z \otimes k_w \in S_1(H)$. Since $M_\varphi^*(k_z) = \overline{\varphi(z)}k_z$, we have

$$C_{M_\varphi^*, M_\psi}(k_z \otimes k_w) = \overline{\varphi(z)}\psi(w)(k_z \otimes k_w).$$

Thus $k_z \otimes k_w$ is an eigen vector for $C_{M_\varphi^*, M_\psi}$ corresponding to the eigen value $\overline{\varphi(z)}\psi(w)$. Now by Lemma 5.2, $\text{span} \{k_z \otimes k_w : \overline{\varphi(z)}\psi(w) \in S^1 \setminus D\}$ is dense in $S_1(H^\beta(\mathbb{D}))$ for any countable set $D \subset S^1$. Hence $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic on $(S_1(H^\beta(\mathbb{D})), \|\cdot\|_1)$ by Proposition 5.1. Since the embeddings $S_1(H^\beta(\mathbb{D})) \hookrightarrow S_p(H^\beta(\mathbb{D})) \hookrightarrow \mathcal{K}(H^\beta(\mathbb{D})) \hookrightarrow (\mathcal{L}(H^\beta(\mathbb{D})), COT)$ are continuous and have dense range, it follows that $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic on each of the these spaces. \square

As noted in the beginning of this section, the Hardy space $H^2(\mathbb{D})$ is a special case of $H^\beta(\mathbb{D})$. In this case, we have the following characterization for $C_{M_\varphi^*, M_\psi}$ on $S_p(H^2(\mathbb{D}))$ and $\mathcal{K}(H^2(\mathbb{D}))$.

Theorem 5.4. *Let φ and ψ be non-zero, bounded and analytic on \mathbb{D} , with one of them being non-constant. Then $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic on $(S_p(H^2(\mathbb{D})), \|\cdot\|_p)$, $(\mathcal{K}(H^2(\mathbb{D})), \|\cdot\|_{op})$ and $(\mathcal{L}(H^2(\mathbb{D})), COT)$ if $|\varphi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$. Conversely, if $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic on $(\mathcal{K}(H^2(\mathbb{D})), \|\cdot\|_{op})$ or $(S_p(H^2(\mathbb{D})), \|\cdot\|_p)$, then $|\varphi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$.*

Proof. We know that $M_\varphi \in \mathcal{L}(H^2(\mathbb{D}))$ if and only if φ is bounded on \mathbb{D} . Thus by the preceding theorem, if $|\varphi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$, then $C_{M_\varphi^*, M_\psi}$ is frequently hypercyclic.

Let $G = \varphi(\mathbb{D})\psi(\mathbb{D})$. Assume that the converse is not true, i.e., $G \cap S^1 = \phi$. Then $G \subseteq \mathbb{D}$ or $G \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$ since G is non-empty and open, where $\overline{\mathbb{D}}$ is the closed unit disc in \mathbb{C} . In case $G \subseteq \mathbb{D}$, then $\|\varphi\|_\infty \|\psi\|_\infty = \sup_{z \in \mathbb{D}} |\varphi(z)| \sup_{w \in \mathbb{D}} |\psi(w)| = \sup_{z, w \in \mathbb{D}} |\varphi(z)\psi(w)| \leq 1$, and consequently,

$$\|C_{M_\varphi^*, M_\psi}\| \leq \|M_\varphi^*\| \|M_\psi\| \leq \|\varphi\|_\infty \|\psi\|_\infty \leq 1.$$

In the latter case when $G \subseteq \mathbb{C} \setminus \overline{\mathbb{D}}$, we have $\inf_{z, w \in \mathbb{D}} |\varphi(z)\psi(w)| \geq 1$, and so, $\inf_{z \in \mathbb{D}} |\varphi(z)| > 0$ and $\inf_{w \in \mathbb{D}} |\psi(w)| > 0$. Thus the functions φ^{-1} and ψ^{-1} are bounded and analytic on \mathbb{D} , and the corresponding operators $M_{\varphi^{-1}}$, $M_{\psi^{-1}}$ are bounded on $H^2(\mathbb{D})$. Moreover $(M_\varphi)^{-1} = M_{\varphi^{-1}}$ and $(M_\psi)^{-1} = M_{\psi^{-1}}$. Then we observe that

$$\|(C_{M_\varphi^*, M_\psi})^{-1}\| = \|C_{M_{\varphi^{-1}}^*, M_{\psi^{-1}}}\| \leq \|M_{\varphi^{-1}}^*\| \|M_{\psi^{-1}}\| \leq \|\varphi^{-1}\|_\infty \|\psi^{-1}\|_\infty \leq 1.$$

Thus, in both the cases, the operator $C_{M_\varphi^*, M_\psi}$ is not hypercyclic on $\mathcal{K}(H^2(\mathbb{D}))$. This contradiction proves that $G \cap S^1 \neq \phi$. \square

As a special case, we state below the frequent hypercyclicity of the conjugate operator $C_{M_\varphi^*}$ for the Hardy space $H = H^2(\mathbb{D})$.

Proposition 5.5. *If φ is non-constant and $|\varphi(z)| = 1$ for some $z \in \mathbb{D}$, then the conjugate map $C_{M_\varphi^*}$ is frequently hypercyclic on $(S_p(H), \|\cdot\|_p)$, $(\mathcal{K}(H), \|\cdot\|_{op})$ and $(\mathcal{L}(H), COT)$. Conversely, if $C_{M_\varphi^*}$ is frequently hypercyclic on $(\mathcal{K}(H), \|\cdot\|_{op})$, then φ is non-constant and $\varphi(\mathbb{D}) \cap S^1 \neq \phi$.*

Proof. Immediate from the preceding theorem. \square

The above characterization is not true for multiplication operators on all Hilbert spaces of analytic functions, e.g. consider

Example 5.6. Let $\mathcal{H} = \{f(z) = \sum_{n \geq 0} a_n z^n : \|f\|^2 = \sum_{n \geq 0} (n+1)^2 |a_n|^2 < \infty\}$. Then \mathcal{H} is a reproducing kernel Hilbert space and the multiplication operator M_φ corresponding to $\varphi(z) = z$ acts as the shift

$$M_\varphi(e_n)(z) = ze_n(z) = \frac{1}{n+1} z^{n+1} = \frac{n+2}{n+1} e_{n+1}(z)$$

with respect to the orthonormal basis $e_n(z) = \frac{1}{n+1} z^n$, $n \geq 0$. Now $w_1 w_2 \dots w_n = n+1$ implies that $\sum_n \frac{1}{(w_1 w_2 \dots w_n)^2} < \infty$ and consequently, the adjoint M_φ^* satisfies the FHC Criterion on \mathcal{H} . Hence by Corollary 3.9, $C_{M_\varphi^*}$ is frequently hypercyclic on the spaces $(S_p(H), \|\cdot\|_p)$, $(\mathcal{K}(\mathcal{H}), \|\cdot\|_{op})$ and $(\mathcal{L}(\mathcal{H}), COT)$. However there are no $z, w \in \mathbb{D}$ such that $|\varphi(z)\varphi(w)| = 1$.

In the spirit of Theorem 5.3, let us prove a similar result about $C_{\phi(B), \psi(F)}$ defined on spaces of operators on ℓ^p , $1 < p < \infty$, where $\varphi(B)$ and $\varphi(F)$ are functions of the unweighted backward and forward shifts respectively. If $\varphi(z) = \sum_{n \geq 0} a_n z^n$ is an analytic function on some neighborhood of the closed disc $\overline{\mathbb{D}}$, then $\varphi(B) = \sum_n a_n B^n$ and $\varphi(F) = \sum_n a_n F^n$ are bounded operators on ℓ^p , $1 \leq p < \infty$. Moreover, the Banach space adjoint of $\varphi(B)$ is $\phi(F)$ and that of $\varphi(F)$ is $\varphi(B)$. Note that if $f_\lambda = (1, \lambda, \lambda^2, \dots)$ and $|\lambda| < 1$, then $\varphi(B)f_\lambda = \varphi(\lambda)f_\lambda$. In [9], R. Delaubenfels and H. Emamirad proved that if $\varphi(\mathbb{D}) \cap S^1 \neq \phi$, then $\varphi(B)$ is hypercyclic on ℓ^p . We now have the following result, which can be proved using Proposition 5.1.

Proposition 5.7. *If φ is non-constant and $\varphi(\mathbb{D}) \cap S^1 \neq \emptyset$, then $\varphi(B)$ is frequently hypercyclic on ℓ^p , $1 \leq p < \infty$.*

Let $\mathcal{N}(\ell^p)$ denote the space of all nuclear operators on ℓ^p . Then the trace $tr(T) = \sum_n x_n^*(x_n)$ of $T = \sum_n x_n \otimes x_n^* \in \mathcal{N}(\ell^p)$, $1 < p < \infty$. Then the dual of $\mathcal{N}(\ell^p)$ is identified with $\mathcal{L}(\ell^p)$ via the trace-duality $(S, T) = tr(TS)$, where $T \in \mathcal{N}(\ell^p)$ and $S \in \mathcal{L}(\ell^p)$, cf. [11], Theorem 16.50. The one-rank operator $f_\lambda \otimes f_\mu$ on ℓ^p is given by $x \rightarrow f_\mu(x)f_\lambda$.

Lemma 5.8. *Let φ and ψ be non-zero functions analytic on some neighborhoods of the closed disc $\overline{\mathbb{D}}$ with one of them being non-constant and $|\varphi(\lambda)\psi(\mu)| = 1$ for some $\lambda, \mu \in \mathbb{D}$. Then*

$$\text{span}\{f_\lambda \otimes f_\mu : \varphi(\lambda)\psi(\mu) \in S^1 \setminus D\}$$

is dense in $\mathcal{N}(\ell^p)$ for every countable set $D \subset S^1$ and $1 < p < \infty$.

Proof. Invoking the proof of Lemma 5.2, we can find an arc $\Gamma \subseteq S^1 \cap \varphi(\mathbb{D})\psi(\mathbb{D})$. Write $U \times V = \{(\lambda, \mu) \in \mathbb{D} \times \mathbb{D} : \varphi(\lambda)\psi(\mu) \in \Gamma \setminus D\}$. Then, for each $\mu \in V$, there exists an uncountable set $U_1 \subset \mathbb{D}$ such that for $\lambda \in U_1$ we have $\varphi(\lambda)\psi(\mu) \in \Gamma \setminus D$.

Let us now prove that $\Delta = \text{span}\{f_\lambda \otimes f_\mu : \lambda \in U, \mu \in V\}$ is dense in $\mathcal{N}(\ell^p)$. For this, let $S \in \mathcal{L}(\ell^p)$ such that $tr(TS) = 0$ for all $T \in \Delta$. In particular, if $T = f_\lambda \otimes f_\mu$ for $\lambda \in U_1$, then $tr(f_\lambda \otimes S^* f_\mu) = (S^* f_\mu)(f_\lambda) = 0$. Since U_1 has limit points in \mathbb{D} , we have that $\text{span}\{f_\lambda : \lambda \in U_1\}$ is dense in ℓ^p . Thus $S^*(f_\mu) = 0$ for all $\mu \in V$. As $\text{span}\{f_\lambda : \mu \in V\}$ is dense in ℓ^{p^*} , the dual of ℓ^p , $S = 0$. \square

Using the above lemma, we prove the frequent hypercyclicity of $C_{\varphi(B), \psi(F)}$ as follows.

Theorem 5.9. *Suppose ϕ and ψ are non-zero analytic maps on some neighborhoods of the closed disc $\overline{\mathbb{D}}$ such that $|\phi(z)\psi(w)| = 1$ for some $z, w \in \mathbb{D}$, with one of ϕ and ψ being non-constant. Then $C_{\phi(B), \psi(F)}$ is frequently hypercyclic on $(\mathcal{N}(\ell^p), \|\cdot\|_{nu})$, $(\mathcal{K}(\ell^p), \|\cdot\|_{op})$ and $(\mathcal{L}(\ell^p), COT)$ for $1 < p < \infty$.*

Proof. Let $\lambda, \mu \in \mathbb{D}$. Since $\phi(B)(f_\lambda) = \phi(\lambda)f_\lambda$, we get

$$C_{\phi(B), \psi(F)}(f_\lambda \otimes f_\mu) = \varphi(B)(f_\lambda) \otimes \psi(B)(f_\mu) = \phi(\lambda)\psi(\mu)(f_\lambda \otimes f_\mu),$$

where $f_\lambda = (1, \lambda, \lambda^2, \dots)$. From Lemma 5.8, it follows that $\text{span}\{f_\lambda \otimes f_\mu : \varphi(\lambda)\psi(\mu) \in S^1 \setminus D\}$ is dense in $\mathcal{N}(\ell^p)$ and the proof is complete by Proposition 5.1. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIAN INSTITUTE OF TECHNOLOGY KANPUR, 208 016 KANPUR, INDIA

E-mail address: manjul@iitk.ac.in

DEPARTMENT OF MATHEMATICS AND STATISTICS, INDIAN INSTITUTE OF TECHNOLOGY KANPUR, 208 016 KANPUR, INDIA

E-mail address: aneeshm@iitk.ac.in